PARABOLIC GEOMETRIES
FOR PEOPLE THAT LIKE PICTURES

LECTURE 7: THE CANONICAL FILTRATION

JACOB W. ERICKSON

Let us first recall the definition of parabolic subgroups from last time.

Definition 0.1. For a subspace $V \subseteq \mathfrak{g}$, its *Killing perp* is the subspace
$$V^\perp := \{X \in \mathfrak{g} : \mathfrak{k}(X, v) = 0 \text{ for each } v \in V\}.$$

Definition 0.2. A subalgebra $\mathfrak{p} \leq \mathfrak{g}$ is *parabolic* if and only if $\mathfrak{p}^\perp$ is a nilpotent subalgebra. A *parabolic subgroup* $P \leq G$, then, is a closed subgroup whose Lie subalgebra $\mathfrak{p}$ is parabolic.

In the last lecture, we spent considerable effort to introduce and motivate these parabolic subgroups in a directly geometric way, as (finite-index subgroups of) stabilizers of points at infinity for a model (G, K). Toward the end, we showed that the Lie algebras of such stabilizers satisfy the above algebraic condition. This time, we will verify that these notions of parabolicity are essentially the same. Along the way, we will introduce some incredibly useful tools from representation theory, including:

- A filtration of a semisimple Lie algebra $\mathfrak{g}$ canonically associated to a parabolic subalgebra $\mathfrak{p}$
- An automorphism θ, called a *Cartan involution*, that swaps horospherical subalgebras
- A grading of a semisimple Lie algebra $\mathfrak{g}$ underlying the canonical filtration

Next time, we will see how these tools interact with the geometry of a model (G, P), where G is semisimple and P is parabolic. In particular, we will be able to get a vague picture of the shape of a general parabolic model geometry.

1. A few examples

For a bit of amusement, it is perhaps worth noting that $O(2) < I(2)$ technically satisfies our definition of parabolicity, since $\mathfrak{o}(2)^\perp$ is the abelian (hence nilpotent) subalgebra of translations. However, while we *can* define parabolic subgroups for arbitrary Lie groups, most would consider the idea of parabolic subgroups to be specific to semisimple Lie groups, for which the Killing form is nondegenerate. Henceforth, we will focus on semisimple model groups.

In $\mathrm{SL}_2\,\mathbb{R}$, recall that we had a closed subgroup B, which we called a *Borel subgroup*[1], defined by

$$B := \left\{ \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} : a \in \mathbb{R}^\times, b \in \mathbb{R} \right\},$$

with Lie subalgebra

$$\mathfrak{b} := \left\{ \begin{bmatrix} a & b \\ 0 & -a \end{bmatrix} : a, b \in \mathbb{R} \right\}.$$

Further, recall that the Killing form on $\mathfrak{sl}_2\mathbb{R}$ is given by

$$\natural\left(\begin{bmatrix} a & b \\ c & -a \end{bmatrix}, \begin{bmatrix} z & y \\ x & -z \end{bmatrix}\right) = 8az + 4(bx + cy).$$

Thus,

$$\natural\left(\begin{bmatrix} a & b \\ 0 & -a \end{bmatrix}, \begin{bmatrix} z & y \\ x & -z \end{bmatrix}\right) = 8az + 4bx,$$

which vanishes for all $\begin{bmatrix} a & b \\ 0 & -a \end{bmatrix} \in \mathfrak{b}$ if and only if $x = z = 0$; in other words, $\mathfrak{b}^\perp = \left\langle \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\rangle =: \mathfrak{b}_+$. Since a 1-dimensional subalgebra is necessarily abelian, this shows that $\mathfrak{b}^\perp$ is a nilpotent subalgebra, hence $\mathfrak{b}$ is parabolic.

Similarly, in $\mathrm{SL}_3\,\mathbb{R}$, we can define a Borel subgroup B as

$$B := \left\{ \begin{bmatrix} r & p & q \\ 0 & s & u \\ 0 & 0 & (rs)^{-1} \end{bmatrix} : r, s \in \mathbb{R}^\times, p, q, u \in \mathbb{R} \right\},$$

the subgroup of upper triangular matrices, with corresponding Lie subalgebra

$$\mathfrak{b} := \left\{ \begin{bmatrix} r & p & q \\ 0 & s & u \\ 0 & 0 & -(r+s) \end{bmatrix} : r, s, p, q, u \in \mathbb{R} \right\}.$$

The Killing form on $\mathfrak{sl}_3\mathbb{R}$ is given by $\natural(R, S) = 6\,\mathrm{tr}(RS)$, where the elements $R, S \in \mathfrak{sl}_3\mathbb{R}$ are considered as linear endomorphisms of $\mathbb{R}^3$ under the "usual" representation of $\mathrm{SL}_3\,\mathbb{R}$. Thus,

$$\begin{aligned}
\natural\left(\begin{bmatrix} r & p & q \\ 0 & s & u \\ 0 & 0 & -(r+s) \end{bmatrix}, \begin{bmatrix} m & a & b \\ x & n & c \\ z & y & -(m+n) \end{bmatrix}\right) &= 6\Big(rm + px + qz + sn + uy \\
&\quad + (r+s)(m+n)\Big) \\
&= 6\Big(r(2m+n) + s(m+2n) \\
&\quad + px + qz + uy\Big),
\end{aligned}$$

---

[1]For a real semisimple Lie group, the term "Borel subgroup" refers to either an arbitrary minimal parabolic subgroup or a specific type of minimal parabolic subgroup that complexifies in a particularly nice way. I used to believe that the former was the better interpretation, but I'm reading more stuff by representation theorists working over $\mathbb{Q}$ and $\mathbb{C}$, and now I'm not sure. Here, the usage of the term is correct regardless of which definition we choose.

which vanishes for every $r, s, p, q, u \in \mathbb{R}$ if and only if $x = y = z = 0$ and $2m + n = m + 2n = 0$, which means that $m = n = 0$ as well. Thus,

$$\mathfrak{b}^{\perp} = \left\{ \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} : a, b, c \in \mathbb{R} \right\},$$

the nilpotent subalgebra of strictly upper triangular matrices, and $\mathfrak{b}$ is parabolic.

For now, the important thing to note is that we get a $\mathfrak{b}$-invariant filtration[2]

$$\mathfrak{sl}_3\mathbb{R} = \mathfrak{g}^{-2} \supset \mathfrak{g}^{-1} \supset \cdots \supset \mathfrak{g}^2 \supset \{0\}$$

of $\mathfrak{sl}_3\mathbb{R}$ given by

$$\mathfrak{g}^{-1} := \left\{ \begin{bmatrix} m & a & b \\ x & n & c \\ 0 & y & -(m+n) \end{bmatrix} : m, n, a, b, c, x, y \in \mathbb{R} \in \mathbb{R} \right\},$$

$$\mathfrak{g}^0 := \mathfrak{b} = \left\{ \begin{bmatrix} m & a & b \\ 0 & n & c \\ 0 & 0 & -(m+n) \end{bmatrix} : m, n, a, b, c \in \mathbb{R} \in \mathbb{R} \right\},$$

$$\mathfrak{g}^1 := \mathfrak{b}^{\perp} = \left\{ \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} : a, b, c \in \mathbb{R} \in \mathbb{R} \right\},$$

$$\mathfrak{g}^2 := [\mathfrak{b}^{\perp}, \mathfrak{b}^{\perp}] = \left\langle \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\rangle.$$

For each i and j, one can show that $[\mathfrak{g}^i, \mathfrak{g}^j] \subseteq \mathfrak{g}^{i+j}$. As it turns out, every parabolic subalgebra has such a filtration canonically associated to it.

2. The canonical filtration

We can generalize our observations from $\mathfrak{b} < \mathfrak{sl}_3\mathbb{R}$ to arbitrary parabolic subalgebras by using the following theorem.

Theorem 2.1. *For $\mathfrak{p}$ a parabolic subalgebra of a semisimple Lie algebra $\mathfrak{g}$, we get a canonical filtration*

$$\mathfrak{g} = \mathfrak{g}^{-k} \supset \mathfrak{g}^{-k+1} \supset \cdots \supset \mathfrak{g}^k \supset \{0\}$$

of $\mathfrak{g}$, defined by $\mathfrak{g}^0 = \mathfrak{p}$, $\mathfrak{g}^1 = \mathfrak{p}^{\perp}$, $\mathfrak{g}^i = [\mathfrak{p}^{\perp}, \mathfrak{g}^{i-1}]$ for each $i > 1$, and $\mathfrak{g}^{-j} = (\mathfrak{g}^{j+1})^{\perp}$ for all j, such that $[\mathfrak{g}^i, \mathfrak{g}^j] \subseteq \mathfrak{g}^{i+j}$ for all i and j.

Proof. The basic idea is to first show that the subspaces $\mathfrak{g}^i$ satisfy $[\mathfrak{g}^i, \mathfrak{g}^j] \subseteq \mathfrak{g}^{i+j}$ without showing that they give a filtration, and then use this to verify that we get a filtration.

For $j > 0$, $[\mathfrak{p}^{\perp}, \mathfrak{g}^j] = \mathfrak{g}^{j+1}$ by definition, and since the Killing form satisfies $\mathfrak{h}([X, Y], Z) = \mathfrak{h}(X, [Y, Z])$, we get $[\mathfrak{p}^{\perp}, \mathfrak{g}^j] \subseteq \mathfrak{g}^{j+1} = (\mathfrak{g}^{-j})^{\perp}$ for

---

[2]The weird choice of direction for these filtrations bothered me at first too; there's a very good reason for it, though, so just go with it.

$j \leq 0$ as well. Thus, by the Jacobi identity, $[\mathfrak{g}^i, \mathfrak{g}^j] \subseteq \mathfrak{g}^{i+j}$ whenever $i > 0$. Using this and the invariance of $\mathring{\flat}$ again, it follows that

$$[\mathfrak{g}^{-i}, \mathfrak{g}^{-j}] = [(\mathfrak{g}^{i+1})^\perp, (\mathfrak{g}^{j+1})^\perp] \subseteq \mathfrak{g}^{-i-j} = (\mathfrak{g}^{i+j+1})^\perp$$

for $i, j \geq 0$ as well, so $[\mathfrak{g}^i, \mathfrak{g}^j] \subseteq \mathfrak{g}^{i+j}$ for all i and j.

Then, to show that $\mathfrak{g}^i \supseteq \mathfrak{g}^{i+1}$ for all i, note that

$$[\mathfrak{g}^{1-i}, [\mathfrak{g}^{i+1}, \mathfrak{g}^j]] \subseteq [\mathfrak{g}^{1-i}, \mathfrak{g}^{i+j+1}] \subseteq \mathfrak{g}^{j+2},$$

so $\mathrm{ad}_X \circ \mathrm{ad}_Y$ is nilpotent for each $X \in \mathfrak{g}^{1-i}$ and $Y \in \mathfrak{g}^{i+1}$, hence we get $\mathfrak{g}^{i+1} \subseteq (\mathfrak{g}^{1-i})^\perp = \mathfrak{g}^i$. $\quad \square$

Since $[\mathfrak{p}^\perp, \mathfrak{p}] \subseteq \mathfrak{p}^\perp$, the subalgebra $\mathfrak{p}^\perp$ is a nilpotent ideal of $\mathfrak{p}$. Moreover, since $[\mathfrak{p}^\perp, \mathfrak{g}^j] \subseteq \mathfrak{g}^{j+1}$, every element of $\mathfrak{p}^\perp$ is ad-nilpotent for $\mathfrak{g}$. This nilpotent ideal $\mathfrak{p}^\perp$ is precisely the horospherical subalgebra $\mathfrak{p}_+$ of $\mathfrak{p}$ that we discussed last time. To define the other horospherical subalgebra $\mathfrak{g}_-$ and the neutral subalgebra $\mathfrak{g}_0$, though, we'll need to define a grading, which will require one more tool from the symmetric space perspective.

3. CARTAN INVOLUTIONS

Given a maximal compact subgroup $K \leq G$, recall from last time that we can decompose $\mathfrak{g}$ (as a vector space) as $\mathfrak{k}^\perp + \mathfrak{k}$, where the Killing form is positive-definite on $\mathfrak{k}^\perp$ and negative-definite on $\mathfrak{k}$. Using this, we can specify a remarkable linear endomorphism θ of $\mathfrak{g}$, called a *Cartan involution*, by defining $\theta|_{\mathfrak{k}^\perp} = -\mathrm{id}_{\mathfrak{k}^\perp}$ and $\theta|_{\mathfrak{k}} = \mathrm{id}_{\mathfrak{k}}$; visually, this just corresponds to reversing geodesic trajectories through e.
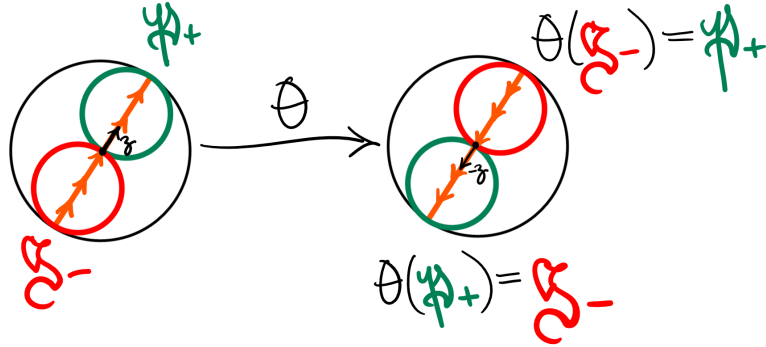

FIGURE 1. The Cartan involution θ reverses geodesic trajectories through e and swaps the horospherical subalgebras associated to each $Z \in \mathfrak{k}^\perp$

The map θ has several useful properties for representation theory. Perhaps chief among these useful properties is that θ happens to be an automorphism of $\mathfrak{g}$, hence an isometry for $\mathring{\flat}$. From its definition, we can also see that $\theta^2 = \mathrm{id}_{\mathfrak{g}}$. Using the decomposition $\mathfrak{k}^\perp + \mathfrak{k}$, we can even see that the symmetric bilinear form $\mathring{\flat}_\theta$ given by $\mathring{\flat}_\theta := \mathring{\flat}(\theta(X), Y)$

is negative-definite on all of $\mathfrak{g}$, which allows us to define things like orthogonal projections.

For now, our main interest in the Cartan involution θ associated to K comes from its behavior on horospherical subalgebras. To see what this behavior is, imagine that we have a point $\gamma(+\infty)$ at infinity, where we can once again take γ to be of the form $t \mapsto q_K(\exp(tZ))$ for some $Z \in \mathfrak{k}^\perp$. For Y an eigenvector of ad_Z with eigenvalue λ,

$$[Z, \theta(Y)] = [-\theta(Z), \theta(Y)] = -\theta([Z,Y]) = -\theta(\lambda Y) = -\lambda\theta(Y),$$

so $\theta(Y)$ is an eigenvector of ad_Z with eigenvalue $-\lambda$. In particular, the Cartan involution θ swaps the horospherical subalgebras $\mathfrak{g}_-$ and $\mathfrak{p}_+$ that we defined last time.

4. Gradings and a fixed point at infinity

With a given Cartan involution θ, we can use the canonical filtration $\mathfrak{g}^i$ to construct an underlying grading

$$\mathfrak{g} = \mathfrak{g}_{-k} + \mathfrak{g}_{-k+1} + \cdots + \mathfrak{g}_k$$

given by $\mathfrak{g}_i := \mathfrak{g}^i \cap \theta(\mathfrak{g}^{-i})$.

Theorem 4.1. *The grading $\sum_i \mathfrak{g}_i$ satisfies the following properties.*
  (1) *For each i, $\theta(\mathfrak{g}_i) = \mathfrak{g}_{-i}$.*
  (2) *For each i, $\mathfrak{g}^i = \sum_{j \geq i} \mathfrak{g}_j$.*
  (3) *For each i and j, $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$.*

Proof. For (1), note that $\theta(\mathfrak{g}_i) = \theta(\mathfrak{g}^i) \cap \theta^2(\mathfrak{g}^{-i}) = \theta(\mathfrak{g}^i) \cap \mathfrak{g}^{-i} = \mathfrak{g}_{-i}$.

For (2), let $\mathfrak{g}^k$ be the smallest nonzero filtration component. Then, $\mathfrak{g}^{k+1} = \{0\}$, so $\mathfrak{g}^{-k} = (\mathfrak{g}^{k+1})^\perp = \mathfrak{g}$, so $\mathfrak{g}_k = \mathfrak{g}^k \cap \theta(\mathfrak{g}^{-k}) = \mathfrak{g}^k$. Proceeding by induction, suppose $\mathfrak{g}^{i+1} = \sum_{j \geq i+1} \mathfrak{g}_j$; we want to prove that

$$\mathfrak{g}^i = \sum_{j \geq i} \mathfrak{g}_j = \mathfrak{g}_i + \mathfrak{g}^{i+1}.$$

To do this, let $\pi : \mathfrak{g}^i \to \mathfrak{g}^{i+1}$ be the $\flat_\theta$-orthogonal projection map. Then, we can decompose each $X \in \mathfrak{g}^i$ as $X = (X - \pi(X)) + \pi(X)$, where $\pi(X) \in \mathfrak{g}^{i+1}$ and $X - \pi(X) \in \theta(\mathfrak{g}^{i+1})^\perp = \theta(\mathfrak{g}^{-i})$, so $\mathfrak{g}^i = \mathfrak{g}_i + \mathfrak{g}^{i+1}$.

Finally, for (3), we know that $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq [\mathfrak{g}^i, \mathfrak{g}^j] \subseteq \mathfrak{g}^{i+j}$, and since θ is an automorphism,

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq [\theta(\mathfrak{g}^{-i}), \theta(\mathfrak{g}^{-j})] = \theta([\mathfrak{g}^{-i}, \mathfrak{g}^{-j}]) \subseteq \theta(\mathfrak{g}^{-i-j}).$$

Thus, $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}^{i+j} \cap \theta(\mathfrak{g}^{-i-j}) = \mathfrak{g}_{i+j}$. $\square$

These properties of the grading tell us several useful things. First, $[\mathfrak{g}_0, \mathfrak{g}_0] \subseteq \mathfrak{g}_{0+0} = \mathfrak{g}_0$, so $\mathfrak{g}_0$ is a subalgebra of $\mathfrak{g}$. Indeed, this $\mathfrak{g}_0$ coincides with the neutral subalgebra introduced last time. Similarly, the subspaces $\mathfrak{p}_+ := \sum_{i>0} \mathfrak{g}_i = \mathfrak{g}^1 = \mathfrak{p}^\perp$ and $\mathfrak{g}_- := \sum_{i<0} \mathfrak{g}_i$ are subalgebras, and coincide with the horospherical subalgebras from last time.

Delightfully, this grading can also help us find a point at infinity for the model (G, K) fixed by our parabolic subgroup $P < G$, retrieving the more directly geometric definition we mentioned in the last lecture. To do this, we define the *grading derivation* $\delta_{\mathrm{gr}} : \mathfrak{g} \to \mathfrak{g}$ to be the linear endomorphism given by $\delta_{\mathrm{gr}}(X) = iX$ for each i and each $X \in \mathfrak{g}_i$.

For $X \in \mathfrak{g}_i$ and $Y \in \mathfrak{g}_j$,

$$\begin{aligned} \delta_{\mathrm{gr}}([X, Y]) &= (i + j)[X, Y] = i[X, Y] + j[X, Y] \\ &= [iX, Y] + [X, jY] = [\delta_{\mathrm{gr}}(X), Y] + [X, \delta_{\mathrm{gr}}(Y)]. \end{aligned}$$

This means that δ_{gr} is a *derivation* on $\mathfrak{g}$, meaning a linear map $\delta : \mathfrak{g} \to \mathfrak{g}$ such that $\delta([X, Y]) = [\delta(X), Y] + [X, \delta(Y)]$. The space $\mathrm{Der}(\mathfrak{g})$ of all derivations of $\mathfrak{g}$ is a Lie algebra under the commutator bracket, and contains the image of ad as a subalgebra. For semisimple Lie algebras, it turns out that *all* derivations are of the form ad_X for some $X \in \mathfrak{g}$.

Lemma 4.2. *For semisimple* $\mathfrak{g}$, $\mathrm{Der}(\mathfrak{g}) = \mathrm{ad}_{\mathfrak{g}}$.

Proof. For $\delta \in \mathrm{Der}(\mathfrak{g})$ and $X \in \mathfrak{g}$, $[\delta, \mathrm{ad}_X] = \mathrm{ad}_{\delta(X)}$, so $\mathrm{ad}_{\mathfrak{g}} \trianglelefteq \mathrm{Der}(\mathfrak{g})$. Thus, the Killing form on $\mathrm{Der}(\mathfrak{g})$ restricts to the Killing form on $\mathrm{ad}_{\mathfrak{g}}$. Since the Killing form on $\mathrm{ad}_{\mathfrak{g}} \approx \mathfrak{g}$ is nondegenerate because $\mathfrak{g}$ is semisimple, we get $\mathrm{Der}(\mathfrak{g}) = \mathrm{ad}_{\mathfrak{g}} \oplus \mathrm{ad}_{\mathfrak{g}}^{\perp}$ as a vector space. But for $\delta \in \mathrm{ad}_{\mathfrak{g}}^{\perp}$, this means

$$[\delta, \mathrm{ad}_X] = \mathrm{ad}_{\delta(X)} \in \mathrm{ad}_{\mathfrak{g}} \cap \mathrm{ad}_{\mathfrak{g}}^{\perp} = \{0\}$$

for all $X \in \mathfrak{g}$, so $\delta = 0$ because $\mathfrak{z}(\mathfrak{g}) = \{0\}$. $\square$

In particular, this tells us that the derivation δ_{gr} is of the form $\mathrm{ad}_{E_{\mathrm{gr}}}$ for some $E_{\mathrm{gr}} \in \mathfrak{g}$. We call this element E_{gr} the *grading element* for the grading on $\mathfrak{g}$. By definition, $\mathrm{ad}_{E_{\mathrm{gr}}} = \delta_{\mathrm{gr}}$ is diagonalizable over $\mathbb{Z} < \mathbb{R}$ on $\mathfrak{g}$ and satisfies $\theta \circ \mathrm{ad}_{E_{\mathrm{gr}}} = -\mathrm{ad}_{E_{\mathrm{gr}}} \circ \theta$, so $E_{\mathrm{gr}} \in \mathfrak{k}^{\perp}$.

The geodesic $t \mapsto q_K(\exp(tE_{\mathrm{gr}}))$ generated by E_{gr} determines a point at infinity fixed by P. This follows directly from our next theorem, which is essentially just Theorem 3.1.3 of [2] and whose proof we consider optional for our current endeavor.

Theorem 4.3. *If* P *is a parabolic subgroup, then it is of the form* $Z_P(E_{gr})P_+$, *where* $Z_P(E_{gr}) = \{p \in P : \mathrm{Ad}_p(E_{gr}) = E_{gr}\}$ *and* P_+ *is the connected subgroup generated by* $\mathfrak{p}_+ = \mathfrak{p}^{\perp}$.

Proof. Suppose $p \in P$. The adjoint action Ad_p on $\mathfrak{g}$ preserves the canonical filtration, so it induces an automorphism $\phi_{\mathrm{gr}}(p)$ of the graded Lie algebra associated to the filtration, so that $\mathrm{Ad}_p Y - \phi_{\mathrm{gr}}(p) \cdot Y \in \mathfrak{g}^{i+1}$ for each $Y \in \mathfrak{g}_i$. In particular, our grading element $E_{\mathrm{gr}} \in \mathfrak{g}_0$ satisfies $\mathrm{Ad}_p E_{\mathrm{gr}} - \phi_{\mathrm{gr}}(p) \cdot E_{\mathrm{gr}} \in \mathfrak{g}^1$, so $\mathrm{Ad}_{p^{-1}}(\phi_{\mathrm{gr}}(p) \cdot E_{\mathrm{gr}}) \in E_{\mathrm{gr}} + \mathfrak{g}^1$.

Let Z_1 be the $\mathfrak{g}_1$-component of $\mathrm{Ad}_{p^{-1}}(\phi_{\mathrm{gr}}(p) \cdot E_{\mathrm{gr}})$, so that

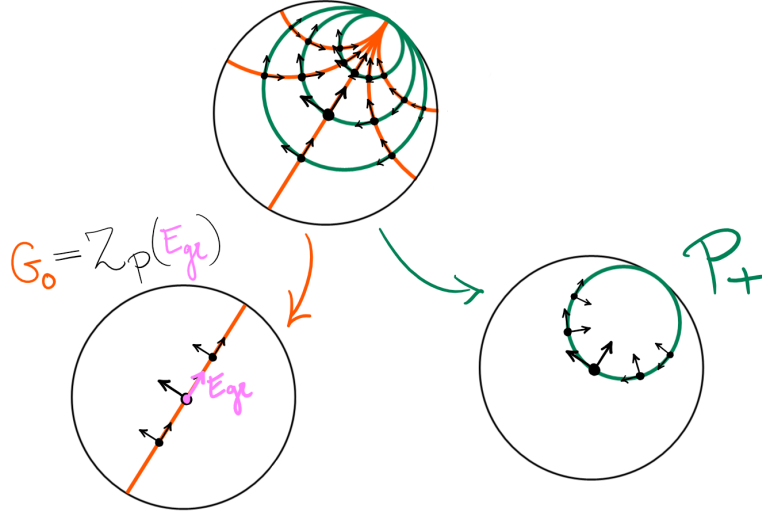$$\mathrm{Ad}_{p^{-1}}(\phi_{\mathrm{gr}}(p) \cdot E_{\mathrm{gr}}) \in E_{\mathrm{gr}} + Z_1 + \mathfrak{g}^2.$$

FIGURE 2. A depiction of a parabolic subgroup, decomposed into G_0, the part that centralizes E_{gr}, and P_+, the part generated by the horospherical subalgebra $\mathfrak{p}_+$

Then, $\mathrm{ad}_{Z_1}(E_{\mathrm{gr}} + Z_1) = -Z_1$ and $\mathrm{ad}_{Z_1}^2(E_{\mathrm{gr}} + Z_1) = 0$, so

$$\mathrm{Ad}_{\exp(Z_1)} \circ \mathrm{Ad}_{p^{-1}}(\phi_{\mathrm{gr}}(p) \cdot E_{\mathrm{gr}}) \in E_{\mathrm{gr}} + \mathfrak{g}^2.$$

Recursively, we define Z_i to be $\frac{1}{i}$ times the $\mathfrak{g}_i$-component of

$$\mathrm{Ad}_{\exp(Z_{i-1})} \circ \cdots \circ \mathrm{Ad}_{\exp(Z_1)} \circ \mathrm{Ad}_{p^{-1}}(\phi_{\mathrm{gr}}(p) \cdot E_{\mathrm{gr}}) \in E_{\mathrm{gr}} + \mathfrak{g}^i,$$

so that

$$\mathrm{Ad}_{\exp(Z_i)} \circ \mathrm{Ad}_{\exp(Z_{i-1})} \circ \cdots \circ \mathrm{Ad}_{\exp(Z_1)} \circ \mathrm{Ad}_{p^{-1}}(\phi_{\mathrm{gr}}(p) \cdot E_{\mathrm{gr}}) \in E_{\mathrm{gr}} + \mathfrak{g}^{i+1}.$$

Eventually, there is some k such that $\mathfrak{g}^{k+1} = \{0\}$, so that

$$\mathrm{Ad}_{\exp(Z_k)} \circ \cdots \circ \mathrm{Ad}_{\exp(Z_1)} \circ \mathrm{Ad}_{p^{-1}}(\phi_{\mathrm{gr}}(p) \cdot E_{\mathrm{gr}}) = E_{\mathrm{gr}},$$

hence $\phi_{\mathrm{gr}}(p) \cdot E_{\mathrm{gr}} = \mathrm{Ad}_{p \exp(-Z_1) \cdots \exp(-Z_k)}(E_{\mathrm{gr}})$. But, recall that $\phi_{\mathrm{gr}}(p)$ is an automorphism of the graded Lie algebra, so that

$$[\phi_{\mathrm{gr}}(p) \cdot E_{\mathrm{gr}}, \phi_{\mathrm{gr}}(p) \cdot Y] = \phi_{\mathrm{gr}}(p) \cdot [E_{\mathrm{gr}}, Y] = i \, \phi_{\mathrm{gr}}(p) \cdot Y$$

for each $Y \in \mathfrak{g}_i$. In particular, $\phi_{\mathrm{gr}}(p) \cdot E_{\mathrm{gr}}$ must agree with the grading element because E_{gr} is the unique element with $\mathrm{ad}_{E_{\mathrm{gr}}} = \delta_{\mathrm{gr}}$. Thus,

$$\phi_{\mathrm{gr}}(p) \cdot E_{\mathrm{gr}} = \mathrm{Ad}_{p \exp(-Z_1) \cdots \exp(-Z_k)}(E_{\mathrm{gr}}) = E_{\mathrm{gr}},$$

hence $p \exp(-Z_1) \cdots \exp(-Z_k) \in \mathrm{Z}_P(E_{\mathrm{gr}})$. $\quad \square$

5. Parabolic model geometries

As one might guess, we can now define a model to be *parabolic* when its model group is semisimple and its isotropy is parabolic.

Definition 5.1. We say that a model geometry (G, P) is *parabolic* when G is a semisimple Lie group and P is a parabolic subgroup.

These parabolic model geometries are the core of the study of parabolic geometries. With our current pacing through the course, we probably won't get to talk much about the general "curved" case, but this was always meant to be more of an invitation to the topic anyway. Next time, we will investigate what these parabolic models look like, in general.

References

[1] Calderbank, D. M. J., Noppakaew, P.: *Parabolic subalgebras, parabolic buildings and parabolic projection.* (2017) arXiv:1607.00370

[2] Čap, A., Slovák, J.: *Parabolic Geometries I: Background and General Theory.* Mathematical Surveys and Monographs, vol. 154. Amer. Math. Soc., Providence, RI (2009)